

GRADE 8 · GEOMETRY · ANSWER KEY

WHY PYTHAGORAS WORKS

Today's goal: I can explain a proof of the Pythagorean Theorem and its converse.


How to use this key. Problem numbers match the student sheets. Each answer shows the steps and the slip Mr DJ sees most, so a wrong answer tells you what to reteach. Pepper levels: **Mild** warm-up, **Medium** stretch, **Spicy** challenge, **Extra** big brain.

Practice

9 problems

Use: In-class practice after the deck. Item 4 is the most valuable and the least obvious: legs of 5 and 5 give $c^2 = 50$, so the hypotenuse is irrational — and that is precisely why the proof works in squares rather than lengths. A child who writes 7.07 has rounded away the exactness the argument depends on. Item 6 is the standard's target in one sentence. Item 8 is worth reading aloud: several children will offer a statement whose converse is also true, and checking the backward direction is the skill. **Timing:** 15–20 min · pairs · items 4, 6 and 8 on the board.

1 $a^2 + b^2 = c^2$, where c is the hypotenuse — the side opposite the right angle, and the longest

2 big square 529, four triangles 240, leftover $289 = 17^2$ **3** not right-angled — $4 + 9 = 13$, and $4^2 = 16$

4 leftover 50, so the hypotenuse squared is 50 and the hypotenuse is irrational

5 $(a + b)^2 = a^2 + 2ab + b^2$; subtract the four triangles' $2ab$; the leftover is $a^2 + b^2$, and that leftover is c^2

6 because no number appears, so the argument holds for every right triangle at once, while a numerical version only settles the triangle it used

7 if $a^2 + b^2 = c^2$ then the triangle is right-angled — it is what lets you TEST whether an angle is right

8 for example: if it is raining then the ground is wet — but wet ground does not mean it has rained

9 she used 5 as a leg and 4 as the hypotenuse — $9 + 25 = 34$, which is not 16

1 **MILD** The theorem

$a^2 + b^2 = c^2$, where c is the hypotenuse — the side opposite the right angle, and the longest

- The two legs are squared and added; the hypotenuse squared is the total.
- Naming c precisely is what stops the sides being used in the wrong places.

Watch for: Stated the equation without saying which side is which. The equation is not true for an arbitrary labelling, so the naming is part of the statement.

2 **MEDIUM** Square count for 8 and 15

big square 529, four triangles 240, leftover $289 = 17^2$

- $(8 + 15)^2 = 23^2 = 529$.
- Four triangles: $4 \times \frac{1}{2} \times 8 \times 15 = 240$.
- $529 - 240 = 289$, and $17^2 = 289$.
- Check: $8^2 + 15^2 = 64 + 225 = 289$. Agreed.

Watch for: Forgot the half and got $289 - \dots$ a leftover of 49. The four triangles form two full rectangles, so their total is $2ab$.

3 **MEDIUM** 2-3-4

not right-angled — $4 + 9 = 13$, and $4^2 = 16$

- Two shorter sides: $2^2 + 3^2 = 13$.
- Longest side squared: 16.
- $13 \neq 16$, so no right angle. This is the converse being used as a test.

Watch for: Compared the wrong pair, or concluded “nearly”. There is no nearly: either the relationship holds or the angle is not 90° .

4 **SPICY** Legs 5 and 5

leftover 50, so the hypotenuse squared is 50 and the hypotenuse is irrational

1. $(5 + 5)^2 = 100$. Four triangles: $4 \times 12.5 = 50$. Leftover: 50.
2. So $c^2 = 50$, and $5^2 + 5^2 = 50$ confirms it.
3. 50 is not a perfect square, so c is not a whole number — it is the square root of 50, which is irrational.
4. This is why the proof is done in SQUARES: the square is a whole number even when the length is not, so nothing has to be rounded.

Watch for: Gave c as about 7.07 and stopped. The decimal is a rounded value, and the exact statement is $c^2 = 50$ — which is the form the proof uses and the reason it works for every triangle, not only the ones with whole-number sides.

5 **SPICY** The proof in letters

$(a + b)^2 = a^2 + 2ab + b^2$; subtract the four triangles' $2ab$; the leftover is $a^2 + b^2$, and that leftover is c^2

1. Big square, side $a + b$, area $(a + b)^2 = a^2 + 2ab + b^2$.
2. Four triangles, each $\frac{1}{2}ab$, total $2ab$.
3. Leftover: $a^2 + 2ab + b^2 - 2ab = a^2 + b^2$.
4. The leftover is a square whose side is the hypotenuse, so its area is c^2 . Hence $a^2 + b^2 = c^2$.

Watch for: Wrote the conclusion without the expansion. The $2ab$ appearing and cancelling is the whole mechanism, so the middle line is the part that matters.

6 **SPICY** Why letters prove it

because no number appears, so the argument holds for every right triangle at once, while a numerical version only settles the triangle it used

1. A calculation with 3 and 4 establishes a fact about that triangle.
2. There are infinitely many right triangles, so no list of calculations can cover them.
3. In letters, a and b stand for ANY legs, and every step is valid whatever they are.
4. So one argument replaces infinitely many calculations — which is what a proof is for.

Watch for: Said letters are “more general” without saying why that matters. The infinitude is the reason: the gap between three cases and all cases cannot be closed by adding cases.

7 **SPICY** The converse

if $a^2 + b^2 = c^2$ then the triangle is right-angled — it is what lets you TEST whether an angle is right

1. The theorem starts from a right angle and concludes something about the sides.
2. The converse starts from the sides and concludes something about the angle.
3. Almost every practical use is the converse: you can measure sides far more easily than you can confirm an angle is exactly 90° .
4. This is how a builder checks a corner is square with a tape measure.

Watch for: Wrote the theorem again. Swapping which part is assumed and which is concluded is what makes it the converse.

8 **SPICY** A true statement with a false converse

for example: if it is raining then the ground is wet — but wet ground does not mean it has rained

1. The forward direction is reliable; the backward one has other explanations, such as a sprinkler.
2. So a statement being true says nothing about its converse.
3. Which is why the converse of Pythagoras needs its own justification rather than following automatically.
4. It happens to be true, and that had to be shown.

Watch for: Gave a statement whose converse is ALSO true, such as “if a triangle is equilateral then all its angles are equal” — that one runs both ways, so it does not show what the item asks for. Test the backwards direction before offering an example.

9 **EXTRA SPICY** Ana's error

she used 5 as a leg and 4 as the hypotenuse — $9 + 25 = 34$, which is not 16

1. $3^2 + 5^2 = 9 + 25 = 34$, while $4^2 = 16$. The statement is false.
2. c must be the hypotenuse, which is the longest side, so $c = 5$ and not 4.
3. Correctly: $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.
4. The equation is not a shape you can pour any three numbers into — each letter names a particular side.

Watch for: Said she “mixed up the numbers”. Naming which side c must be is the answer, and showing $34 \neq 16$ is what makes it a demonstration rather than an assertion.

Homework

8 problems

Use: Homework after the practice sheet. Item 6 is the one to mark first, and the distinction is sharp: the problem is not that a hundred is a small sample — a million would fail identically — but that no finite list exhausts an infinite collection. Item 2 asks for the hypotenuse SQUARED deliberately, since $c^2 = 40$ is exact and 6.32 is not. Item 4 is close enough to tempt a rounding, and $61 \neq 64$ is the whole answer. **Timing:** 20–25 min · independent · items 6, 4 and 8 first when marking.

1 big square 961, four triangles 336, leftover 625, hypotenuse 25

2 big square 64, four triangles 24, leftover 40, so $c^2 = 40$

3 yes — $81 + 1600 = 1681 = 41^2$. This uses the converse

4 not right-angled — $25 + 36 = 61$, and $8^2 = 64$

5 $(a + b)^2 = a^2 + 2ab + b^2$; the four triangles total $2ab$; subtracting leaves $a^2 + b^2$, which is the square on the hypotenuse, so $a^2 + b^2 = c^2$

6 because there are infinitely many right triangles, so any finite number of checks leaves infinitely many unchecked

7 converse: if a number is divisible by 5 then it ends in 0. That is FALSE — 15 is divisible by 5 and ends in 5

8 because the proof shows the theorem is true for every right triangle rather than the ones checked, and it explains WHY the areas match — which is what lets it be used and trusted in unfamiliar cases

1 MEDIUM Legs 7 and 24

big square 961, four triangles 336, leftover 625, hypotenuse 25

1. $(7 + 24)^2 = 31^2 = 961$.
2. $4 \times \frac{1}{2} \times 7 \times 24 = 336$.
3. $961 - 336 = 625$, and $25^2 = 625$, so the hypotenuse is 25.
4. Check: $49 + 576 = 625$.

Watch for: Forgot the half. Two full rectangles, so $2ab = 336$.

2 MEDIUM Legs 2 and 6

big square 64, four triangles 24, leftover 40, so $c^2 = 40$

1. $(2 + 6)^2 = 64$. Four triangles: $4 \times 6 = 24$. Leftover: 40.
2. Check: $4 + 36 = 40$.
3. 40 is not a perfect square, so the hypotenuse is irrational — $c^2 = 40$ is the exact answer.

Watch for: Gave c as 6.32. The question asks for the hypotenuse SQUARED, which is exactly 40.

3 MEDIUM 9-40-41

yes — $81 + 1600 = 1681 = 41^2$. This uses the converse

1. $9^2 + 40^2 = 81 + 1600 = 1681$.
2. $41^2 = 1681$.
3. Equal, so the triangle is right-angled. Going from sides to angle is the converse.

Watch for: Named the theorem. Sides in, angle out, is the converse.

4 MEDIUM 5-6-8

not right-angled — $25 + 36 = 61$, and $8^2 = 64$

1. Two shorter sides: $25 + 36 = 61$.
2. Longest squared: 64.
3. $61 \neq 64$, so no right angle — close, and close is not equal.

Watch for: Rounded and said yes. 61 and 64 are different numbers; the angle is near 90° but not 90° .

5 SPICY The proof in letters

$(a + b)^2 = a^2 + 2ab + b^2$; the four triangles total $2ab$; subtracting leaves $a^2 + b^2$, which is the square on the hypotenuse, so $a^2 + b^2 = c^2$

1. Expand the big square: $a^2 + 2ab + b^2$.
2. Four triangles at $\frac{1}{2}ab$ each: $2ab$.
3. $a^2 + 2ab + b^2 - 2ab = a^2 + b^2$.
4. The leftover region is a square of side c , so its area is c^2 .

Watch for: Skipped the expansion. The cancelling $2ab$ is the mechanism, so it has to appear.

6 SPICY Why a hundred is not enough

because there are infinitely many right triangles, so any finite number of checks leaves infinitely many unchecked

1. A hundred cases is a hundred true statements about a hundred triangles.
2. The claim is about all of them, and there is no largest right triangle to finish at.
3. Legs can be any positive numbers at all, including irrational ones, so they cannot even be listed.
4. One argument in letters covers the whole infinite collection, which is the only way to close the gap.

Watch for: Said a hundred is “not many”. The issue is not the size of the sample — a million would fail for the same reason. It is that no finite list can exhaust an infinite collection.

7 **SPICY** A converse about divisibility

converse: if a number is divisible by 5 then it ends in 0. That is FALSE — 15 is divisible by 5 and ends in 5

1. Swap the two halves to form the converse.
2. A counterexample settles it: 15.
3. So here is a true statement with a false converse, which is why a converse always needs checking.
4. Pythagoras' converse is true, and that is a separate fact from the theorem.

Watch for: Said the converse is true because both are about 5. One counterexample is all that is needed, and 15 is the easiest.

8 **EXTRA SPICY** Why a proof

because the proof shows the theorem is true for every right triangle rather than the ones checked, and it explains WHY the areas match — which is what lets it be used and trusted in unfamiliar cases

1. A rule you cannot justify is one you cannot check when a calculation looks wrong.
2. The proof also shows the theorem is about AREAS of squares, which is why the relationship involves squaring at all — that connection is invisible if it is only a formula.
3. It makes the converse a question you would think to ask, instead of a second formula.
4. And it is the difference between believing the theorem and knowing it.

Watch for: Said “so you understand it”. The concrete answers are: it covers every case, and it reveals that the squaring comes from actual squares.

Exit Ticket

4 problems

Use: End of the proof lesson. Item 3 is the diagnostic and it is the standard's actual target: a child who says “check more examples” has restated the misconception rather than diagnosed it, because more cases still leave infinitely many unchecked. The answer wanted is the algebra, and specifically the observation that no number appears in it. Item 4 tests the direction of the implication — almost every practical use is the converse, and children reach for “the theorem” because that is the name they know. **Timing:** 5–8 min · independent · squared paper helps.

1 big square 289, four triangles 120, leftover 169

2 $169 = 13^2$, and 13 is the hypotenuse

3 he has checked three cases out of infinitely many — the algebra is what proves it

4 no — $36 + 49 = 85$ and $8^2 = 64$, so the relationship fails. This uses the CONVERSE

1 **MEDIUM** The square count

big square 289, four triangles 120, leftover 169

1. Big square: $(5 + 12)^2 = 17^2 = 289$.
2. One triangle: $\frac{1}{2} \times 5 \times 12 = 30$. Four of them: 120.
3. Leftover: $289 - 120 = 169$.

Watch for: Used 5×12 as the triangle's area, forgetting the half, giving 240 and a leftover of 49. The four triangles make two rectangles, so $2ab$ is the total — a useful check.

2 **MEDIUM** The leftover is c squared

$169 = 13^2$, and 13 is the hypotenuse

1. $13^2 = 169$, which is the leftover area.
2. So the square on the hypotenuse equals the big square minus the four triangles.
3. Equivalently $5^2 + 12^2 = 25 + 144 = 169$.

Watch for: Stopped at 169 without connecting it to 13. The connection is the whole point of the count.

3 **SPICY** What Tom is missing

he has checked three cases out of infinitely many — the algebra is what proves it

1. Every one of his checks is correct, and there are infinitely many right triangles.
2. Three examples give evidence, and no number of examples covers all of them.
3. The fix is the same square picture written in LETTERS: $(a + b)^2 - 2ab = a^2 + b^2$.
4. Because no number appears in that line, it holds for every right triangle at once — which is what a proof has to do.

Watch for: Said he needs to check more examples. That is the misconception rather than the fix — more cases still leave infinitely many unchecked. Also common: “he needs to explain it”, which is right and does not say what the explanation is.

4 **SPICY** Is 6-7-8 right-angled?

no — $36 + 49 = 85$ and $8^2 = 64$, so the relationship fails. This uses the CONVERSE

1. Take the two shorter sides: $6^2 + 7^2 = 36 + 49 = 85$.
2. The longest side squared: $8^2 = 64$.
3. 85 does not equal 64, so the triangle is not right-angled.
4. Going from the SIDES to a conclusion about the ANGLE is the converse. The theorem runs the other way: it starts from knowing there is a right angle.

Watch for: Said “the theorem”. Almost every practical use of Pythagoras to TEST a triangle is the converse, and noticing which direction you are travelling is what this item is for.

Lesson slides

teaching notes & timing

Slide	Time	What to do
1	0:00	Say clearly that today is not about calculating a missing side. Children expect Pythagoras to be a calculation, and the standard asks for the proof.
2	0:45	State the claim carefully, and pin c to the hypotenuse. Half the later errors are c in the wrong place, and it is cheaper to prevent here.
3	1:30	Build the picture on the board, or with four paper triangles. The corners-out arrangement is the one that leaves a tilted square in the middle.
4	2:45	Count with 3 and 4. Everyone gets a number they believe: $49 - 24 = 25$.
5	4:00	THE slide, and it needs to feel like a reversal. They have just proved it — and they have not. Check two more triples so the evidence piles up, then ask how many right triangles there are. Three out of infinitely many is the honest count.
6	5:30	The same picture in letters. Go slowly through $(a+b)^2 = a^2 + 2ab + b^2$, then subtract the $2ab$. The moment worth naming: no number appears anywhere, which is exactly why it covers all of them.
7	7:00	The converse. Give a non-mathematical example of a statement whose converse fails — raining means wet ground, wet ground does not mean rain. Then say which direction the useful test needs.

9	8:00	The 5-12 count and the letters, then the exit ticket. Whiteboards: everyone tries Mild, then climbs. These are #1, #2, #4 on the practice sheet.
10		Fingers up: 3 = got it, 2 = almost, 1 = not yet. Pull the 1s for a quick reteach while everyone else starts the practice sheet.
11		Launch the practice sheet. Exit ticket in the last 5–8 minutes.